Uniqueness of the Fock quantization of Dirac fields with unitary dynamics

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Motivation

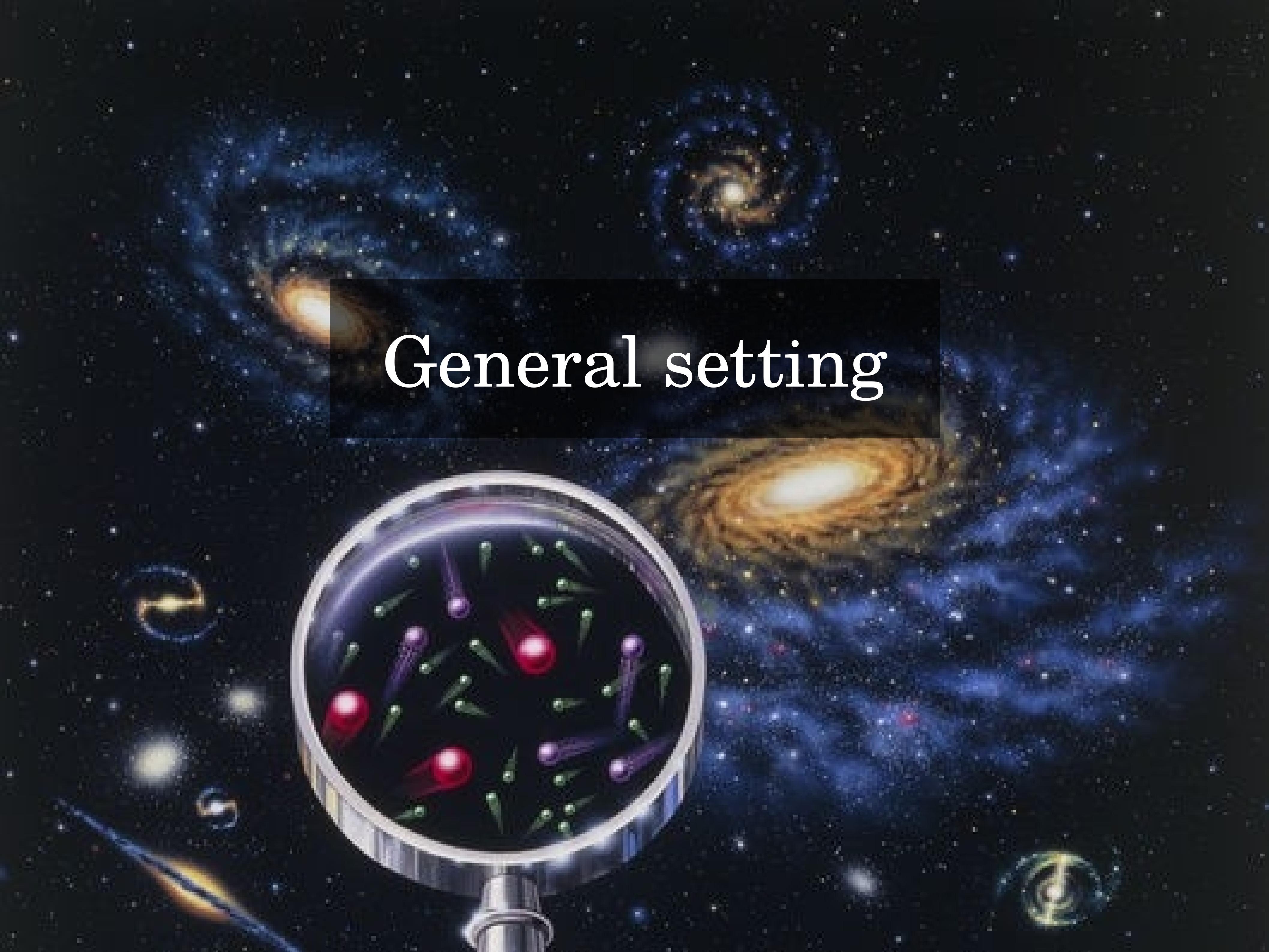
- QFT $\longrightarrow$ Infinite ambiguity when choosing a quantization (infinitely many degrees of freedom, no SvN theorem).

- Minkowski spacetime, e.g., ambiguity cured with symmetry requirements.

- In nonstationary spacetimes, can we impose some physically meaningful criteria...?

Motivation

- QFT $\longrightarrow$ Infinite ambiguity when choosing a quantization (infinitely many degrees of freedom, no SvN theorem).
- Minkowski spacetime, e.g., ambiguity cured with symmetry requirements.
- FRW cosmologies $\longrightarrow$ highly symmetric spatial sections.
- Criteria: Invariance under the isometry group + unitary dynamics.
- Application to realistic fields in nature $\longrightarrow$ The Dirac fermion.


General setting

The Dirac field in curved spacetimes

- Dirac equation on a globally hyperbolic spacetime,

  $S = \{\psi\}$ Linear space of solutions.

- Global hyperbolicity $\Longrightarrow$ $S \approx$ set of data on a Cauchy surface.

- Natural inner product $(\psi_1, \psi_2)_S$, conserved under evolution.

- Analogous construction of $\bar{S}$.

Fermion Complex Structures

- Codify the ambiguity in the choice of Fock representation.

- Real linear map J defined on S and on $\bar{S}$, (equiv. on set of data)

$$J^2 = -I, \qquad (J\psi_1, J\psi_2)_S = (\psi_1, \psi_2)_S$$

- Defines a splitting into its $\pm i$ eigenspaces

$$S_J^{\pm} = \frac{1}{2}(S \mp iJS), \qquad \bar{S}_J^{\pm} = \overline{S_J^{\mp}}$$

$S_J^+ \longrightarrow$ Particle annihilation $\qquad S_J^- \longrightarrow$ Antiparticle creation

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$$J^2 = -I, \qquad (J\psi_1, J\psi_2)_S = (\psi_1, \psi_2)_S$$

- Completion of $S_J^+ \longrightarrow$ 1-p Hilbert space

$$\oplus$$

- Completion of $\overline{S_J^-} \longrightarrow$ 1-ap Hilbert space

$\Rightarrow$ Antisym. Fock sp.

Cosmological model

- Closed FRW cosmology, scale factor $\exp[\alpha(\eta)]$, spatial sections S^3.

- Minimally coupled massive Dirac field, described by

  $\phi_A, \bar{\chi}_{A'}$ $\quad A=1,2;\ A'=1',2',$ $\quad$ Grassmann variables

  local repr. of cross-sections of a spinor bundle, gauge group $SL(2,\mathbb{C})$.

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Partial gauge fixing

Gauge group $SU(2)$

Mode expansion of the fields in terms of a complete set of spinors on S^3.

Cosmological model

$$\phi_A(x) = \frac{e^{-3\alpha(\eta)/2}}{2\pi} \sum_{npq} \breve{\alpha}_n^{pq} [m_{np}(\eta) \rho_A^{nq}(\vec{x}) + \bar{r}_{np}(\eta) \bar{\sigma}_A^{nq}(\vec{x})]$$

$$\bar{\chi}_{A'}(x) = \frac{e^{-3\alpha(\eta)/2}}{2\pi} \sum_{npq} \breve{\beta}_n^{pq} [\bar{s}_{np}(\eta) \bar{\rho}_{A'}^{nq}(\vec{x}) + t_{np}(\eta) \sigma_{A'}^{nq}(\vec{x})]$$

$$n \in \mathbb{N}; \qquad p, q = 1, \ldots, g_n; \qquad g_n = (n+1)(n+2)$$

- Eigenspinors of the Dirac operator on S^3:

$\rho_A^{np}, \sigma_{A'}^{np} \longrightarrow$ Eigenvalue $+\omega_n$, deg. g_n

$\bar{\rho}_{A'}^{np}, \bar{\sigma}_A^{np} \longrightarrow$ Eigenvalue $-\omega_n$, deg. g_n

$$\boxed{\omega_n = n + \frac{3}{2}}$$

- Complete set for the expansion of any spinor and c.c. on S^3.

Invariant Complex Structures

Representations isometry group

- Isometry group $SO(4) \Longleftrightarrow \mathrm{Spin}(4)$ via Clifford multiplication.
- Active $SO(4)$ transformations on $S^3 \Rightarrow$ Passive $\mathrm{Spin}(4)$ transformations on the spinors of a given chirality.
- Unitary representation: Direct sum of irreps. of $\mathrm{Spin}(4)$.
- Each Dirac op. eigenspace $\longrightarrow$ each inequivalent irrep. of $\mathrm{Spin}(4)$.
- Parallel and component-wise equivalent decomposition for c.c.

Representations isometry group

$$\phi_A(x)=\frac{e^{-3\alpha(\eta)/2}}{2\pi}\sum_{npq}\breve{\alpha}_n^{pq}[m_{np}(\eta)\rho_A^{nq}(\vec{x})+\bar{r}_{np}(\eta)\bar{\sigma}_A^{nq}(\vec{x})]$$

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$$n\in\mathbb{N}; \qquad p,q=1,\ldots,g_n; \qquad g_n=(n+1)(n+2)$$

- Irreducible decomposition of $\mathrm{Spin}(4)$:

$\{\rho_A^{np}\}_p,\ \{\bar{\rho}_{A'}^{np}\}_p \longrightarrow$ Two equivalent irreps.

$\{\bar{\sigma}_A^{np}\}_p,\ \{\sigma_{A'}^{np}\}_p \longrightarrow$ Two equivalent irreps.

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$\{\rho_A^{np}\}_p, \{\bar{\rho}_{A'}^{np}\}_p \longrightarrow$ Two equivalent irreps.

Inequivalent

$\{\bar{\sigma}_A^{np}\}_p, \{\sigma_{A'}^{np}\}_p \longrightarrow$ Two equivalent irreps.

Invariant complex structures

$$\phi_A(x)=\frac{e^{-3\alpha(\eta)/2}}{2\pi}\sum_{npq}\breve{\alpha}_n^{pq}\left[m_{np}(\eta)\rho_A^{nq}(\vec{x})+\bar{r}_{np}(\eta)\bar{\sigma}_A^{nq}(\vec{x})\right]$$

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- Schur's lemmas
  - Invariant J cannot mix modes with $\neq n$.
  - It cannot mix $\{m_{np},\bar{s}_{np}\}$ with $\{t_{np},\bar{r}_{np}\}$.
  - It can only mix pairs of modes with same p.
  - Maps that mix $(m_{np},\bar{s}_{np})$ do not depend on p.
  - Maps that mix $(t_{np},\bar{r}_{np})$ do not depend on p.

Invariant complex structures

- Invariant complex structures select, at each time,

$$\begin{pmatrix} a_{np}^{(x,y)} \\ b_{np}^{(x,y)\dagger} \\ a_{np}^{(x,y)\dagger} \\ b_{np}^{(x,y)} \end{pmatrix}_\eta = \begin{pmatrix} f_1^n(\eta) & f_2^n(\eta) & 0 & 0 \\ g_1^n(\eta) & g_2^n(\eta) & 0 & 0 \\ 0 & 0 & \bar{f}_1^n(\eta) & \bar{f}_2^n(\eta) \\ 0 & 0 & \bar{g}_1^n(\eta) & \bar{g}_2^n(\eta) \end{pmatrix} \begin{pmatrix} x_{np} \\ \bar{y}_{np} \\ \bar{x}_{np} \\ y_{np} \end{pmatrix}_\eta , \quad (x_{np}, y_{np}) = \begin{cases} (m_{np}, s_{np}) \\ (t_{np}, r_{np}) \end{cases}$$

$$|f_1^n|^2 + |f_2^n|^2 = 1, \qquad |g_1^n|^2 + |g_2^n|^2 = 1, \qquad f_1^n \bar{g}_1^n + f_2^n \bar{g}_2^n = 0,$$

so they are annihilation and creation-like.

Time-dep. phase.

$$g_1^n = \bar{f}_2^n e^{iG^n}, \qquad g_2^n = -\bar{f}_1^n e^{iG^n},$$

Unitary dynamics

Fermion dynamics

$$(x_{np}, y_{np}) = \begin{cases} (m_{np}, s_{np}) \\ (t_{np}, r_{np}) \end{cases}$$

- First order Dirac equations

$$x_{np}' = i\omega_n x_{np} - ime^{\alpha}\bar{y}_{np}, \qquad y_{np}' = i\omega_n y_{np} + ime^{\alpha}\bar{x}_{np}, \qquad ' := \frac{d}{d\eta}$$

- Same second order equation for all $\{z_{np}\} := \{x_{np}, y_{np}\}$

$$z_{np}'' = \alpha' z_{np}' - (\omega_n^2 + m^2 e^{2\alpha} + i\omega_n \alpha') z_{np}$$

Fermion dynamics

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Mass couples the two chiralities!

- First order Dirac equations

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Fermion dynamics: asymptotics

- 1st order Dirac eqs. $\Longrightarrow$ Same 2nd order eq. for $\{z_{np}\} := \{x_{np}, y_{np}\}$.

- Two independent solutions $\exp[i\Theta_n^1(\eta)]$, $\exp[-i\Theta_n^2(\eta)]$,

$$\Theta_n^l(\eta) = \omega_n \Delta\eta + \frac{i}{2}[1 + (-1)^l]\Delta\alpha + \int_{\eta_0}^{\eta} d\tilde{\eta}\, \Lambda_n^l(\tilde{\eta}), \qquad l = 1,2$$

$$\Lambda_n^l(\eta) = O(\omega_n^{-1}), \quad \Delta\eta = \eta - \eta_0, \quad \Delta\alpha = \alpha - \alpha_0.$$

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$$\Lambda_n^l(\eta) = O(\omega_n^{-1}), \quad \Delta\eta = \eta - \eta_0, \quad \Delta\alpha = \alpha - \alpha_0.$$

- Using the 1st order Dirac equations:

$$\boxed{\begin{aligned} x_{np}(\eta) &= e^{i\Theta_n^1(\eta)} x_{np}^0 - \Gamma_n\left[e^{i\Theta_n^1(\eta)} - e^{-i\Theta_n^2(\eta)}\right]\bar{y}_{np}^0, \\ y_{np}(\eta) &= e^{i\Theta_n^1(\eta)} y_{np}^0 + \Gamma_n\left[e^{i\Theta_n^1(\eta)} - e^{-i\Theta_n^2(\eta)}\right]\bar{x}_{np}^0, \end{aligned}}$$

$$\Gamma_n = \frac{m e^{\alpha_0}}{2\omega_n + i\alpha'_0}$$

Dynamical transformation

- The fermion dymanics induces

$$\begin{pmatrix} a_{np}^{(x,y)} \\ b_{np}^{(x,y)\dagger} \end{pmatrix}_{\eta} = B_n(\eta,\eta_0) \begin{pmatrix} a_{np}^{(x,y)} \\ b_{np}^{(x,y)\dagger} \end{pmatrix}_{\eta_0}, \qquad B_n = \begin{pmatrix} \alpha_n^f & \beta_n^f \\ \beta_n^g & \alpha_n^g \end{pmatrix},$$

and c.c., with $|\beta_n^h(\eta,\eta_0)|$ given by

$$\Big| \left[-h_1^n \left(h_2^{n,0} + \Gamma_n h_1^{n,0} \right) e^{i\int \Lambda_n^1} + \bar{\Gamma}_n h_2^n h_2^{n,0} e^{\Delta\alpha} e^{i\int \bar{\Lambda}_n^2} \right] e^{i\omega_n \Delta\eta} + \\ + \left[h_2^n \left(h_1^{n,0} - \bar{\Gamma}_n h_2^{n,0} \right) e^{-i\int \bar{\Lambda}_n^1} + \Gamma_n h_1^n h_1^{n,0} e^{\Delta\alpha} e^{-i\int \Lambda_n^2} \right] e^{-i\omega_n \Delta\eta} \Big|$$

denoting $h = f, g$.

- There is a symmetry: $|\beta_n^f(\eta,\eta_0)| = |\beta_n^g(\eta,\eta_0)|$.

Conditions for unitary dynamics

- The Bogoliubov transformation defined by the sequence $B_n(\eta,\eta_0)$ is unitarily implementable in the Fock space if and only if

$$\sum_n g_n |\beta_n^f(\eta,\eta_0)|^2 < \infty, \qquad \sum_n g_n |\beta_n^g(\eta,\eta_0)|^2 < \infty, \qquad \forall\, \eta.$$

- Recall: $g_n = (n+1)(n+2) = \omega_n^2 - \dfrac{1}{4}.$

- Unitarily implementable evolution implies as a ***necessary*** condition:

$$\boxed{\beta_n^h(\eta,\eta_0) \ \text{ negligible with respect to } \ \omega_n^{-1}}$$

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- This condition requires a specific behavior for $f_1^n, f_2^n, g_1^n, g_2^n$ both in their dependence on ω_n and on η.

Conditions for unitary dynamics

$$\boxed{\beta_n^h(\eta,\eta_0) \text{ negligible with respect to } \omega_n^{-1}} \longrightarrow o(\omega_n^{-1})$$

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$$h_l^n = (-1)^{l+1}\frac{me^{\alpha}}{2\omega_n}e^{iH_{\tilde{l}}^n} + o(\omega_n^{-1}), \qquad \{l,\tilde{l}\} := \{1,2\} \text{ as a set}$$

$H_{\tilde{l}}^n$ phase of $h_{\tilde{l}}^n$

for all $n \in \mathbb{N}_l$, with $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$ (up to a finite subset).

- Unless...

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Conditions for unitary dynamics

- Necessary condition

$$h_l^n = (-1)^{l+1} \frac{me^{\alpha}}{2\omega_n} e^{iH_{\tilde{l}}^n} + \vartheta_{h,l}^n, \qquad n \in \mathbb{N}_l, \qquad \vartheta_{h,l}^n \sim o(\omega_n^{-1})$$

- Introducing in $|\beta_n^h(\eta, \eta_0)|$ and recalling $g_n \sim O(\omega_n^2)$, a non-trivial unitary dynamics is attained if and only if

$$\sum_{n \in \mathbb{N}_l} g_n |\vartheta_{h,l}^n|^2 < \infty, \qquad \forall \eta, \qquad l = 1,2$$

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- The possible non-square summable terms in $\vartheta_{h,l}^n$ have a time dependence that trivializes the dynamics of the fermion modes, at the studied order.

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- Introducing in $\left|\beta_n^h(\eta,\eta_0)\right|$ and recalling $g_n \sim O\left(\omega_n^2\right)$, a non-trivial unitary dynamics is attained if and only if

$$\sum_{n \in \mathbb{N}_l} g_n \left|\vartheta^n_{h,l}\right|^2 < \infty, \qquad \forall \eta, \qquad l = 1,2$$

- ~~The possible non-square summable terms in $\vartheta^n_{h,l}$ have a time dependence that trivializes the dynamics of the fermion modes, at the studied order.~~

Non-trivial unitary dynamics

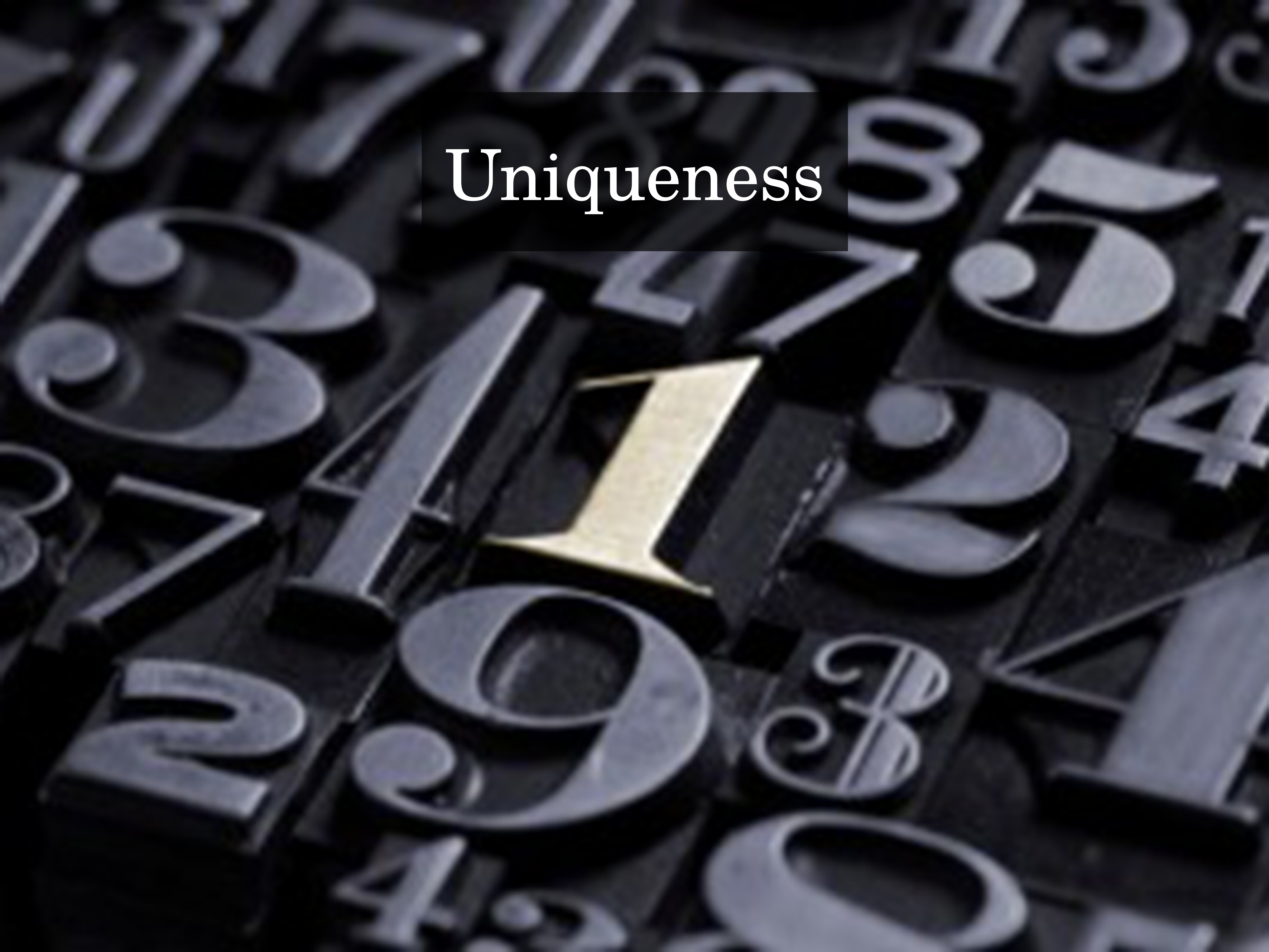
Invariant Fock quantization of the massive Dirac field in a closed FRW cosmology admits a non-trivial and unitary dynamics

$$\Updownarrow$$

$$\boxed{h_l^n = (-1)^{l+1} \frac{m e^{\alpha}}{2\omega_n} e^{iH^n_{\tilde{l}}} + \vartheta^n_{h,l},} \qquad \boxed{\sum_{n\in\mathbb{N}_l} g_n \left|\vartheta^n_{h,l}\right|^2 < \infty, \quad \forall\, \eta,}$$

$$n \in \mathbb{N}_l, \qquad \{l, \tilde{l}\} = \{1,2\}$$

Uniqueness



Reference quantization

- Reference J_R: Simplest choice of invariant complex structure that admits a (non-trivial) unitary quantum dynamics

$$f_1^n = \frac{me^{\alpha}}{2\omega_n}, \qquad f_2^n = \sqrt{1-(f_1^n)^2}, \qquad g_1^n = f_2^n, \qquad g_2^n = -f_1^n,$$

both for (m_{np}, s_{np}) and for (t_{np}, r_{np}).

Unitary equivalence

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- Its relation with any other invariant $\tilde{J}$ given by

$$\begin{pmatrix} \tilde{a}_{np}^{(x,y)} \\ \tilde{b}_{np}^{(x,y)\dagger} \end{pmatrix}_{\eta} = K_n(\eta) \begin{pmatrix} a_{np}^{(x,y)} \\ b_{np}^{(x,y)\dagger} \end{pmatrix}_{\eta}, \qquad K_n = \begin{pmatrix} \kappa_n^f & \lambda_n^f \\ \lambda_n^g & \kappa_n^g \end{pmatrix},$$

$$\lambda_n^h = \frac{\tilde{h}_1^n h_2^n - \tilde{h}_2^n h_1^n}{h_2^n k_1^n - h_1^n k_2^n}, \qquad \{h, k\} := \{f, g\} \text{ as a set.}$$

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- The Bogoliubov transformation defined by the sequence $K_n(\eta)$ is unitarily implementable in the Fock space if and only if

$$\sum_n g_n |\lambda_n^f(\eta)|^2 < \infty, \qquad \sum_n g_n |\lambda_n^g(\eta)|^2 < \infty, \qquad \forall\, \eta.$$

$$\lambda_n^h = \frac{\tilde{h}_1^n h_2^n - \tilde{h}_2^n h_1^n}{h_2^n k_1^n - h_1^n k_2^n}.$$

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- Take $\tilde{J}$ to admit a non-trivial unitarily implementable dynamics

$$\tilde{f}_l^n = \frac{me^{\alpha}}{2\omega_n} e^{i\tilde{F}_{\tilde{l}}^n} + \vartheta_{\tilde{f},l}^n, \quad n \in \mathbb{N}_l, \quad \{l, \tilde{l}\} = \{1,2\}, \quad \sum_{n\in\mathbb{N}_l} g_n |\vartheta_{\tilde{f},l}^n|^2 < \infty.$$

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- For $n \in \mathbb{N}_1$,

$$\lambda_n^f = \vartheta_{\tilde{f},1}^n + O(\omega_n^{-2}), \qquad \lambda_n^g = -\bar{\vartheta}_{\tilde{f},1}^n e^{i\tilde{G}^n} + O(\omega_n^{-2})$$

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- For $n \in \mathbb{N}_2$, λ_n^f, λ_n^g are of order unity $\longrightarrow$ not s.q.s!

- However...

Uniqueness

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- For $n \in \mathbb{N}_2$, λ_n^f, λ_n^g are of order unity $\longrightarrow$ not s.q.s!

- Can be understood as due to a reversal in the convention of particles and antiparticles for an infinite collection of modes.

Uniqueness

- Reference $\widetilde{J}_R$ same as J_R for $n \in \mathbb{N}_1$, but

  $f_1^n \leftrightarrow g_1^n$, $f_2^n \leftrightarrow g_2^n$, i.e., particles $\leftrightarrow$ antiparticles, $n_2 \geq n \in \mathbb{N}_2$.

- Take $\widetilde{J}$ to admit a non-trivial unitarily implementable dynamics

$$\widetilde{f}_l^n = \frac{m e^{\alpha}}{2\omega_n} e^{i \widetilde{F}_{\widetilde{l}}^n} + \vartheta_{\widetilde{f},l}^n, \quad n \in \mathbb{N}_l, \quad \{l, \widetilde{l}\} = \{1,2\}, \quad \sum_{n \in \mathbb{N}_l} g_n \left|\vartheta_{\widetilde{f},l}^n\right|^2 < \infty.$$

- For $n \in \mathbb{N}_1$,

$$\lambda_n^f = \vartheta_{\widetilde{f},1}^n + O\left(\omega_n^{-2}\right), \qquad \lambda_n^g = -\bar{\vartheta}_{\widetilde{f},1}^n e^{i\widetilde{G}^n} + O\left(\omega_n^{-2}\right)$$

- For $n \in \mathbb{N}_2$,

$$\lambda_n^f = \vartheta_{\widetilde{f},2}^n + O\left(\omega_n^{-2}\right), \qquad \lambda_n^g = \bar{\vartheta}_{\widetilde{f},2}^n e^{i\widetilde{G}^n} + O\left(\omega_n^{-2}\right)$$

Uniqueness

- Reference $\tilde{J}_R$ same as J_R for $n \in \mathbb{N}_1$, but

  $f_1^n \leftrightarrow g_1^n$, $f_2^n \leftrightarrow g_2^n$, i.e., particles $\leftrightarrow$ antiparticles, $n_2 \geq n \in \mathbb{N}_2$.

- Take $\tilde{J}$ to admit a non-trivial unitarily implementable dynamics

  $$\tilde{f}_l^n = \frac{me^{\alpha}}{2\omega_n} e^{i\tilde{F}_{\tilde{l}}^n} + \vartheta_{\tilde{f},l}^n, \quad n \in \mathbb{N}_l, \quad \{l, \tilde{l}\} = \{1,2\}, \quad \sum_{n \in \mathbb{N}_l} g_n |\vartheta_{\tilde{f},l}^n|^2 < \infty.$$

- For $n \in \mathbb{N}_1$,

  $$\lambda_n^f = \vartheta_{\tilde{f},1}^n + O(\omega_n^{-2}), \qquad \lambda_n^g = -\bar{\vartheta}_{\tilde{f},1}^n e^{i\tilde{G}^n} + O(\omega_n^{-2})$$

- For $n \in \mathbb{N}_2$,

  $$\lambda_n^f = \vartheta_{\tilde{f},2}^n + O(\omega_n^{-2}), \qquad \lambda_n^g = \bar{\vartheta}_{\tilde{f},2}^n e^{i\tilde{G}^n} + O(\omega_n^{-2})$$

Unitary equiv.!

Conclusions

- Combined criteria of invariance of the vacuum under the isometry group of the closed FRW cosmology + unitary implementation of the dynamics $\longrightarrow$ unique Fock quantization of the Dirac field.

- Uniqueness attained after fixing the convention of particles and antiparticles for an infinite number of modes.

- Requirement of unitary dynamics $\longrightarrow$ fixes the parameterization of the dominant parts of the field, through a global time-dependent rescaling, unless one ought to trivialize the field dynamics.

- Possible extension of results to the massless case in $2+1$ dim...